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The Hubble Law and the Spiral Structures of Galaxies from Equations of Motion in General Relativity*

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Abstract

Fully exploiting the Lie group that characterizes the underlying symmetry of general relativity theory, Einstein's tensor formalism factorizes, yielding a generalized (16-component) quaternion field formalism. The associated generalized geodesic equation, taken as the equation of motion of a star, predicts the Hubble law from one approximation for the generally covariant equations of motion, and the spiral structure of galaxies from another approximation. These results depend on the imposition of appropriate boundary conditions. The Hubble law follows when the boundary conditions derive from the oscillating model cosmology, and not from the other cosmological models. The spiral structures of the galaxies follow from the same boundary conditions, but with a different time scale than for the whole universe. The solutions that imply the spiral motion are *Fresnel integrals.* These predict the star's motion to be along the "Cornu Spiral." The part of this spiral in the first quadrant is the imploding phase of the galaxy, corresponding to a motion with continually decreasing radii, approaching the galactic center as time increases. The part of the "Cornu Spiral" in the third quadrant is the exploding phase, corresponding to continually increasing radii, as the star moves out from the hub. The spatial origin in the coordinate system of this curve is the inflection point, where the explosion changes to implosion. The two- (or many-) armed spiral galaxies are explained here in terms of two (or many) distinct explosions occurring at displaced times, in the domain of the rotating, planar galaxy.

1. *Introduction*

The theory of general relativity is based on a tacit assumption that there exists an underlying order for all of the physical manifestations of the universe–from the elementary particle domain to the cosmological domain of the entire universe. The starting hypothesis–*the principle of relativity*–asserts that this

* Dedicated to the memory of Cornelius Lanczos – friend, human being, scholar.

order, as expressed in "laws of nature," must be objective. This implies that the laws of nature, as expressed in terms of the space-time coordinates of frames of reference that are in arbitrary sorts of relative motion, must be in one-to-one correspondence.

There is an implication in the assertion of the principle of relativity that rejects the classical view of the space and time coordinates as "things-in-themselves." They are rather interpreted in the theory of relativity as not more than language elements, whose only purpose is to facilitate an expression of the laws of nature.

Einstein's theory of general relativity is based on the idea that the *logic* of the space-time language is indeed not arbitrary, but it is rather a representation of matter. This is in terms of the mutual interactions of all of the matter content of a closed system (in principle). The logic of the language that characterizes the space-time, analogous to the syntax of ordinary language structure, is made up of two parts. The first logically relates the elements of the continuum of points of space-time, in the sense of *geometry*, that is, in terms of metric relations, relations of congruence, parallelism, and so on. The second part of the logic of space-time is that which underlies the relations of the numbers of a sequence, that are used to identify, combine, and enumerate the points of the space-time. This logic is in the sense of their *algebra*. The complete logic of space-time then requires a specification of the geometry and algebra and the expression of this total logical system as a representation of matter.

While Einstein concentrated in his writings on general relativity theory on the relation of geometry to physics, the relation of algebra to physics was propounded a century earlier by William Rowan Hamilton. In 1853, in his "Preface to Lectures on Quaternions" (Halberstam and Ingram, 1967), Hamilton made the following remarks:

> It early appeared to me · · · to regard ALGEBRA as being no mere Art, nor Language, nor *primarily* as Science of Quantity; but rather as the Science of Order in Progression. It was, however, a part of this conception, that the *progression* here spoken of was understood to be *continuous* and *unidimensional*: extending indefinitely *forward* and *backward*, but not in any *lateral* direction. And although the successive *states* of such a progression might (no doubt) be represented by *points upon a line*, yet I thought that their simple *successiveness* was better conceived by comparing them with *moments of time*, divested, however, of all reference to *cause* and *effect*; so that the "time" here considered might be said to be abstract, ideal, or *pure*, like that "space" which is the object of geometry. In this manner I was led, many years ago, to regard Algebra as the SCIENCE OF PURE TIME.
> · · · And with respect to anything unusual in the *interpretations* thus proposed, · · · it is my wish to be understood as not at all insisting on them as *necessary*, but merely proposing them as consistent

> among themselves, and preparatory to the study of the quaternions, in at least one aspect of the latter.

With the later appearance of the theory of general relativity, in the twentieth century, it is important, in retrospect, to take notice of Hamilton's comment that the "time" he refers to is not the perceptual reaction that one identifies with the duration of a physical system – that is, it is not directly identified with cause–effect relations. The "time" he refers to, as expressed in terms of his Algebra, is rather the "abstract, ideal, or pure" relational concept. This "time" seems to me to correspond closely with the time component of Einstein's abstract space-time, as a relative language construct to express objective laws of nature. The fusion of Hamilton's discoveries about the quaternion algebra and his interpretation of this in terms of an abstract time, with Einstein's relativization of this "time" with space, in the abstract "space-time" of general relativity theory, then leads in a natural way to a generalization of the mathematical representation of the theory of general relativity. This, in turn, leads to additional physical predictions from the formalism of the theory.

Such a fusion of algebra with geometry in the expression of general relativity theory in its irreducible form, then, indeed, leads to the *necessity* (that Hamilton had not yet seen at his stage!) for the use of the quaternion algebra in this expression of laws of nature. This conclusion, to be discussed in Sec. 2, follows from the requirement that the mathematical expression of the theory should transform as an irreducible representation of the underlying symmetry group of general relativity theory.

Since the initial successes of Einstein's theory of general relativity, it has been interpreted by some scholars as a generalized theory of the gravitational force. This is in the sense of superseding Newton's theory of universal gravitation by reproducing all of the successful classical results, as well as making successful predictions of gravitational phenomena that are not predicted by Newton's theory. In terms of its conceptual structure, the theory of general relativity provides a fundamental explanation for the specific inverse-square law of Newton's theory. This is derived from asymptotic features of the solutions of the field equations of Einstein's theory, replacing Newton's *action-at-a-distance* concept, where the mutual interactions of matter propagate at a finite speed.

The primary success of Einstein's field theory has been the explanation of the observed features of the gravitational force in the domain of the solar system, as well as the explanation of terrestrial gravitational effects. Still, the theory of general relativity can hardly yet claim any great success in explaining the astronomical features of the night sky, beyond our solar system, such as the observed properties of the nearby galaxies and our own galaxy, *the Milky Way*. That is to say, if the theory is as general as it is purported to be, it should *predict* the features associated with the stars, galaxies, and clusters of galaxies, such as their distributions in space and their motions, relative to our position as observers in the universe.

Einstein himself did not interpret his field theory as only a theory of the gravitational force. Rather, he looked upon his tensor field equations that had already successfully explained some gravitational phenomena as a preliminary form for a field formalism that should represent a general theory of matter. He had hoped that when properly generalized, the formalism of general relativity theory would include the electromagnetic manifestations as well as the other features of interacting matter in the microscopic domain of elementary particle physics, where (conceptually) the oppositely oriented quantum theory has thus far been evoked to explain the data of atomic and elementary particle physics. Although, historically, Einstein first discovered the theory of special relativity, and then proceeded to the case of general relativity, if one should now accept Einstein's interpretation of his theory as a general theory of matter, then the case of special relativity should not be considered as more than a special asymptotic limit, where matter is sufficiently rarefied to replace an actual curved space-time with its (flat) tangent space-time at each point of observation. It is clear that this approximation has had a great deal of success in providing at least partial explanations for the physical manifestations of matter in the microscopic domain, such as the additional physical implications of imposing special relativistic covariance on the quantum mechanical equations (e.g., the "spin" magnetic moment of an electron), the relativistic Doppler effects, the energy-mass relation, and so on.

If Einstein was right about the interpretation of his field theory as a general theory of matter, then it must follow that (1) the same generally covariant formalism that correctly predicts the features of matter in the microscopic domain of elementary particle physics should also be applicable as a general mathematical representation for the cosmological domain, and (2) the equations of motion of the general field theory should correctly predict the details of the astromoners' observations of the motions of the stars.

In regard to the former requirement, I have shown in earlier publications (Sachs, 1967, 1968, 1970a) that the usual symmetric second rank tensor representation of Einstein's general relativity theory is indeed not its most general mathematical representation. When one proceeds to the most general form of the theory, according to the algebraic structure of the irreducible representations of the underlying symmetry group (the "Einstein group"), one is led to a factorization of the second rank tensor form, where the basic metrical field now becomes a (16-component) four-vector field, in which each of the vector components obeys the algebraic properties of quaternions. The equations that these field variables solve are a set of 16 relations at each space-time point, rather than the 10 relations of the conventional tensor form of the theory. In Sec. 2 I will review the mathematical derivation and structure of the quaternion representation of general relativity theory. In previous publications (Sachs, 1971, 1972a) I have applied this quaternion field representation (which is equivalent to a second rank spinor reprsentation of a special type) to the elementary particle domain. It was found that the formalism could be structured to asymptotically approach the formalism of ordinary quantum mechanics, in the limits as the parameters relating to energy-momentum transfer between interacting components of a system of matter become sufficiently small. Also, explicit field relationships were found relating to the inertial masses of elementary particles. Within the framework of this theory, the masses of the electron and the muon were derived as the members of a mass doublet (Sachs, 1972b), and the muon lifetime was determined within this theory (Sachs, 1972c).

In this paper, attention will be given to the second requirement discussed above, where the formalism of the general quaternion representation of the theory is applied to determine the motions of the stars. Here one would expect that if the theory is to be successful, the equation of motion of a single star, as a "test body" subject to the influence of the matter of the rest of the universe (in principle), is the geodesic equation. According to general relativity theory, the affine connection terms in this equation are the geometrical representation for the rest of the matter of the universe that causes this star to move in the way that it does.

What are the explicit facts that we now know about the motions of the stars that should be predicted by a correct equation of motion? The first observational fact, that most contemporary astronomers accept as conclusive, is the *Hubble law*—the assertion of a linear relation between the speed of a star and its distance from an observer, such as the astronomer here on Earth (Sandage, 1972a). There has been some recent controversy on the possibility of a breakdown of this linear relation for the quasars—the most distantly observed stellar objects (Sandage, 1972b). The latter anomalous results have led some astronomers to speculate that the stars observed near the horizon of the observable universe are in fact slowing down, and that eventually they will stop, turn around and start to move inward in an imploding phase. This implies to these astronomers that perhaps a more valid cosmology than the *single big bang model* would be an oscillating universe (with periodic implosion and explosion for all of the mass of the universe).

I will show in Sec. 3 that starting with the generalized quaternion version of the geodesic equation, and imposing the boundary conditions of a harmonically oscillating universe, the Hubble law follows from the dynamical solutions for this equation of motion. The "field" that plays the role of the "Hubble constant" in this derivation relates to a term in the geodesic equation implied by the quaternion representation of general relativity theory that does not appear in the conventional expression of the geodesic equation. It is also indicated in the explicit form of the "Hubble constant", so derived, how its mathematical deviation from constancy could arise, should such deviation for the motions of the quasars be proven as a conclusive experimental fact.

At this point in the discussion, it should be emphasized that while other well known cosmological models have proven to be successful in *incorporating* the Hubble law, none of these models have actually *derived* this type of motion (that is claimed to be an experimental fact) from an equation of motion. Rather, the procedure had been to set up a metric tensor solution with a particular spatial symmetry and time dependence, specified beforehand so as to incorporate this law. From the view of the theory of general relativity, one of the troubles with this method has been the effect of sacrificing the general covariance of the theory—as it happens, for example, in the emergence of a global time that is the same for all observers (a "global, absolute time") in the Robertson-Walker metric—to yield the Hubble law, (Adler et al., 1965). But it seems to me that the breakdown of general covariance in the imposed solution (that satisfies an altered set of field equations in order to match this solution) does not serve as a bona fide test of the theory of general relativity, although it might represent an adequate approximation for an actually covariant solution of general relativity theory.

With the same equations of motion and boundary conditions that predict the Hubble law from the quaternion formalism, another mathematical approximation applied to the relatively infinitesimal domain, associated with a single galaxy rather than the entire universe, predicts a spiral motion for the star, moving toward the center of the spiral as time progresses in the imploding phase. The implication here is that the oldest stars of spiral galaxies, such as the *red giants*, should be found near the galactic center. The observed spiral structure of some of the galaxies, if explained in this way, is then a superposition of spiral motions of different stars, subject to imploding and exploding cycles within the domain of stellar matter of the observed galaxy. The mathematical solution that predicts the spiral motion from the geodesic equation is the *Fresnel integral*—giving the *Cornu spiral* path.

2. *Review of the Quaternion Representation of General Relativity Theory*

The theory of general relativity is based, axiomatically, on *the principle of relativity*—the assertion that the laws of nature must be totally objective. That is to say, this principle asserts that the laws of nature must be independent of the frame of reference from which they are expressed. If the frame of reference is represented in terms of a language of space and time coordinates, then there must be a unique set of transformations from the space-time coordinates of one reference frame to any other that would preserve the forms of the laws of nature. The starting assumption about these transformations,

$$x^\alpha \to x^{\alpha\prime}(x^0, x^1, x^2, x^3)$$

which distinguish one reference frame from another, is that they are *continuous*. Assuming further, that these transformations are analytic, the necessary and sufficient conditions are provided for the field equations of the theory to incorporate conservation laws *in the local limit*. If we take the latter incorporation to be a physical requirement of the theory, it follows that the laws of nature, generally, must be in the form of field equations whose solutions are analytic functions of the underlying space-time coordinates, and whose covariance is with respect to an underlying Lie group.

The number of essential parameters that characterizes this Lie group is the number of independent transformations $(\partial x^{\alpha\prime}/\partial x^\beta)$ at each space-time point. This is generally equal to 16 for a nonlinear space-time. Thus, the underlying symmetry group of general relativity theory is a 16-parameter Lie group—the

"Einstein group." It was discovered by Einstein that laws of nature are covariant with respect to the same set of space-time coordinate transformations that leave invariant the squared differential increment of a Riemannian space-time,

$$ds^2 = g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta \tag{2.1}$$

The physical reason for imposing the coordinate-dependent relation, $g_{\alpha\beta}(x)$, between the points of space-time was Einstein's contention that the geometry of space-time is not more than a representation of the matter content of the physical system. Thus if the basic representation of matter is in terms of a coordinate-dependent field, then the geometric relations between the points of space-time must, correspondingly, be a continuously variable field. The next step was then to find the explicit relations between the matter fields, on the one hand, and the geometrical field on the other.

Einstein's tensor field equations,

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta} \tag{2.2}$$

were found to relate the matter content of a physical system, represented in terms of the generally covariant tensor, $T_{\alpha\beta}$ (that is the global extension of the energy-momentum tensor that solves the local conservation laws) to the metric tensor field, $g_{\alpha\beta}$, that is involved in the nonlinear differential form $G_{\alpha\beta}$ on the left-hand side of these field equations. Einstein's field equations transform covariantly in a Riemannian space-time as a second rank symmetric tensor representation of the Einstein group. This representation corresponds to the specification of 10 independent relations at each space-time point. But the Einstein group is a 16-parameter Lie group—implying that a full exploitation of such a symmetry must entail 16 relations at each space-time point. It must then be concluded that Einstein's field equations (2.2) are not the maximally general representation of general relativity theory (Sachs, 1970b).

The reason that the field equations (2.2) are not the most general representation of the theory is that in addition to the *continuous* coordinate transformations that *define* general covariance within this theory—because the relative "motion" is defined to be a continuous entity—Einstein's equations (2.2) are *also* covariant with respect to reflections in space and time. One must then remove the reflection symmetry elements in order to yield the most general representation of the theory, in accordance with the irreducible representations of the Einstein group. When this is done, it is found that the four-dimensional real representations of the transformations that leave ds^2 [Eq. (2.1)] invariant, which previously included reflections in space-time, are now decomposed into the direct sum of two two-dimensional complex, Hermitian representations (Einstein and Mayer, 1932). These have the algebraic properties of a quaternion number field.

The implication of this result in general relativity theory is that the quadratic real number form (2.1) for ds^2 should factorize into a product of a quaternion form and its conjugate. Thus, I have investigated the following factorization (Sachs, 1967):

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \to \begin{cases} ds = q_\alpha\, dx^\alpha \\ \tilde{ds} = \tilde{q}_\alpha\, dx^\alpha \end{cases} \tag{2.3}$$

The metrical field, $q_\alpha(x)$, is *geometrically* a four-vector in a Riemannian space-time. However, each of the four components of this vector is, *algebraically*, a quaternion. Thus, $q_\alpha(x)$ is a 16-component field, and the factorized form ds for the invariant metric of the Riemannian space-time is, algebraically, a quaternion, rather than a real number. This is a generalization of the usual representation of the invariant metric of the Riemannian manifold since ds must now be specified by four real numbers at each space-time point, rather than the single real number, $(g_{\alpha\beta}\, dx^\alpha\, dx^\beta)^{1/2}$, according to the usual formulation.

This generalization is analogous to the analytic continuation of a real number metric invariant into the complex plane, if one were to apply complex function analysis to this formulation. In the latter case, the single parameter description of the invariant metric would be extended to a two-parameter description. That is, to *define* a geodesic path in the complex plane, rather than along the real axis, one must specify two real numbers (the real and imaginary parts of s) at each point of this path. Extending further, from the complex space of pairs to the quaternion space of quadruples, as in the present analysis, it is the four-parameter quaternion description of the invariant metric that is indicated for the complete specification of a geodesic path, according to the structure of the irreducible representations of the Lie group that underlies the symmetry of general relativity theory.

Since ds obeys the algebraic properties of a quaternion number field, rather than a real number field, its simplest representation is in terms of a two-dimensional Hermitian matrix,

$$(ds)_{ij} = (ds)^*_{ji}$$

It also follows from this definition that the "path integral" in a Riemannian space-time must similarly be represented, most simply, by a two-dimensional Hermitian matrix,

$$\left(\int_{s_1}^{s_2} q_\alpha\, dx^\alpha \right)_{ij} = \left(\int_{s_1}^{s_2} q_\alpha\, dx^\alpha \right)^*_{ji}$$

where the end points of the integration, s_1, s_2, denote the quadruples of numbers that define the limits of the path integral. In contrast with the path integral of the ordinary Riemannian space-time, which is parametrized by a single real variable, this path integral is parametrized by four real number variables. That is, at each space-time point x, one must use four real numbers in order to proceed in the sequence of points (that define the path) that is ds further along the path. This is indeed an important consequence of the generalization from the real number invariant, $ds = \pm (g_{\alpha\beta}\, dx^\alpha\, dx^\beta)^{1/2}$, to the quaternion invariant, $ds = q_\alpha\, dx^\alpha$.

It is important to emphasize that the (four-parameter) quaternion differential invariant, ds, or the quaternion path integral invariant, $\int_{s_1}^{s_2} ds$, are not directly observable quantities in the transcription of this geometrical formulation into a physical problem, such as the prediction of the path of a moving "test body." But neither are the corresponding quantities of the (single-parameter) conventional geometrical representation of a line element in a Riemannian space-time directly observable physical quantities. The observables, relating to the motion of a "test body," come from the *equations of motion*, which are taken in general relativity theory to be the geodesic equation.

The extra predictions from this generalized quaternion representation of the geometry of space-time in general relativity, show up in (1) the features of the space-time resulting from its "torsion," and (2) the dependence of the quaternion variables on the space-time-dependent variables in phase factors. In this paper, only the second of these new aspects of the formalism will be exploited. A time-dependent phase factor appears in the quaternion metrical fields, when describing a "stationary state." This feature, in turn, leads to an extra term in the geodesic equation for the moving body, when it is expressed in terms of the "time parameter" rather than the invariant s (Sachs, 1970a). This result is entirely analogous to the time-dependent phase factor that appears in the Schrödinger or Dirac solutions, when describing an electron in a stationary state. The extra term that appears in the wave equation in either of the latter theories depends linearly on the electron's energy.

In Sec. 3, the torsional properties of the space-time are projected out by taking the trace of the quaternion form of the geodesic equation. Nevertheless, the torsion is always present in the general form of the equations of motion, and it does imply physical predictions whose explicit forms are not investigated in this paper.

To express the geodesic equation in its quaternion form, quaternion derivatives d/ds must be defined. This was done in a previous article (Sachs, 1970a). The conventional limiting procedure led to the *definition*

$$d/ds = q_\alpha^{-1}\, d/dx^\alpha \tag{2.4}$$

where the inverse quaternion, q_α^{-1}, defined at the space-time point where d/dx^α is applied, is algebraically equal to $\tilde{q}_\alpha/\tilde{q}_\alpha q_\alpha$ (no sum is implied here). The denominator, $\tilde{q}_\alpha q_\alpha$ is the norm of the quaternion q_α.

We see, then, that the quaternion calculus in differential geometry entails the form (2.4) for the quaternion parametric derivative, rather than the real number form, $d/ds = d/(g_{\alpha\beta}\, dx^\alpha\, dx^\beta)^{1/2}$ of the conventional calculus. An important feature of the derivatives of the type (2.4), as compared with the ordinary (real number) derivatives, or the derivatives in a complex plane, is that the result of two quaternion differentiations depends generally on their order of application, that is, they are noncommutative.

The norm of the quaternion invariant ds is the real number

$$|ds\, \tilde{ds}| = \tfrac{1}{2} |q_\alpha \tilde{q}_\beta + q_\beta \tilde{q}_\alpha|\, dx^\alpha\, dx^\beta \tag{2.3'}$$

Thus, the correspondence between the tensor and the quaternion metrical fields is the following:

$$g_{\alpha\beta} \Leftrightarrow \tfrac{1}{2}(q_\alpha \tilde{q}_\beta + q_\beta \tilde{q}_\alpha) \qquad (2.5)$$

The quaternion metrical field itself can be expressed in its two-dimensional Hermitian form as follows:

$$q_\alpha(x) = \sigma_\beta v_\alpha{}^\beta(x) \qquad (2.6)$$

where the quaternion basis elements, $\sigma_\beta = (\sigma_0; \sigma_k)$, are the unit two-dimensional matrix and the three Pauli matrices. These play the role of Hamilton's $(1; i, j, k)$ basis elements of a quaternion. With this algebraic structure, the 16 field components, $v_\alpha{}^\beta$, are defined with norms as follows: $v_0{}^\beta v_{0\beta} = 1$, $v_k{}^\beta v_{k\beta} = -1$ $(\bar{k} = 1, 2, 3)$.

The 16-component field, $v_\alpha{}^\beta$, relates to the tetrad field that has been investigated by other authors in studies in general relativity. It is important to note, however, that the present formulation is not equivalent to the standard tetrad formulation, because the quaternion formalism entails the metric field in terms of a *whole quaternion*, with its added restraints such as noncommutability under multiplication, hermiticity, and so on. This is analogous to the consideration of the solutions of Dirac's electron equation as *whole spinor* variables (rather than their separate components), as the fundamental variables.

It also might be added here, parenthetically, that the factorization (2.3) follows for precisely the same reason that the Klein–Gordon equation in special relativity theory factorizes into a conjugated pair of two-component spinor field equations in Dirac's electron theory. It is because of the removal of the reflection symmetry elements from the underlying Lorentz group, yielding the factorization of the D'Alembertian operator

$$\Box \equiv (\partial^0)^2 - \nabla^2 \rightarrow \begin{cases} \sigma_\beta \partial^\beta \\ \\ \tilde{\sigma}_\beta \partial^\beta \end{cases}$$

The basis functions of the factorized (quaternion) differential operators above are the two-component spinor variables for Dirac's relativistic electron. Similarly, the removal of the reflection symmetry elements from the symmetry group of general relativity theory (the "Einstein group") leads to the quaternion factorization (2.3).

The quaternion representation of the theory of general relativity yields a nonredundant formulation, prescribing 16 (differential) equations and 16 unknowns at each space-time point. Starting from the factorization (2.3), I have derived all of the tensors of a Riemannian manifold in terms of the quaternion variables and their conjugates, as the basic metric field components rather than the metric tensor $g_{\alpha\beta}$ (Sachs, 1967). From a variational calculation, using the Palatini technique, and treating the counterpart of the Riemann scalar curvature $R(q_\alpha, \tilde{q}_\alpha)$ as the part of the Lagrangian density that yields the explicit behavior of the metric field, the equations in q_α were found to take the following form:

$$\tfrac{1}{4}(K_{\alpha\beta}q^{\beta} + q^{\beta}K_{\alpha\beta}) + Rq_{\alpha} = \kappa \mathscr{T}_{\alpha} \qquad (2.7)$$

$R = \frac{1}{4}\mathrm{Tr}\,(K_{\alpha\beta}(q^{\beta}\tilde{q}^{\alpha} - q^{\alpha}q^{\beta}) + \mathrm{H.c.})$ is the explicit form that corresponds to the scalar curvature field of a Riemannian manifold, $K_{\alpha\beta}$ is the spin curvature," defined as follows in terms of the second covariant derivatives of a two-component spinor field variable η:

$$\eta_{;\alpha;\beta} - \eta_{;\beta;\alpha} = K_{\alpha\beta}\eta \equiv (\partial_{\alpha}\Omega_{\beta} + \Omega_{\alpha}\Omega_{\beta} - \partial_{\beta}\Omega_{\alpha} - \Omega_{\beta}\Omega_{\alpha})\eta$$

and Ω_{α} is the "spin affine connection," defined in terms of the covariant derivative of a spinor field,

$$\eta_{;\alpha} = \partial_{\alpha}\eta + \Omega_{\alpha}\eta$$

The vanishing of the covariant derivatives of the quaternion variables q_{α} (taking account of their geometrical four-vector transformation property *and* their algebraic property as a second rank spinor of the form $\eta \otimes \eta^*$) then yields

$$\Omega_{\alpha} = \tfrac{1}{4}(\partial_{\alpha}\tilde{q}^{\beta} + \Gamma^{\beta}_{\gamma\alpha}\,\tilde{q}^{\gamma})\,q_{\beta}$$

In the quaternion field equations (2.7), the matter source field, $\mathscr{T}_{\alpha}$, follows from the variation of the matter Lagrangian density with respect to the quaternion field variables. The latter is the part of the Lagrangian whose variation with respect to the field variables other than the quaternion metrical fields yields the generally covariant particle wave equations, and the other (nonmetrical) field equations.

The quaternion field equations (2.7) are the 16 independent relations at each space-time point x that underlie the explicit features of the metric space. They are fully covariant according to the underlying symmetry group of general relativity theory, and they transform according to the lowest-dimensional irreducible representations of this group. Thus, this is the most general representation of the theory.

When the quaternion field equations (2.7) are iterated with a conjugated quaternion solution, $\tilde{q}_{\beta}$, the reconstructed equations then transform geometrically as a second-rank tensor representation of the Einstein group. While this is not a symmetric or an antisymmetric tensor representation, it can be rewritten as the sum of a symmetric tensor part (10 relations) and an antisymmetric tensor part (six relations) (Sachs, 1968). When this was done, it was found that the symmetric tensor part is in one-to-one correspondence with Einstein's field equations (2.2), in accordance with the correspondence of the tensors of a Riemannian space-time and particular constructions of q_{α}, $\tilde{q}_{\alpha}$ and their derivatives, as determined earlier (Sachs, 1967). Thus, all of the physical predictions of Einstein's formalism are also predicted by the quaternion formalism (2.7). However, because there are six extra (nonredundant) field equations coming out of the iterated version, there must be more physical predictions here than are made by Einstein's field equations.

It is significant that in the iterated form of the quaternion field equations

(2.7), the symmetric and antisymmetric tensor parts have opposite reflection properties in space-time. Where the symmetric tensor part is even under reflections, the antisymmetric tensor part is odd. This result is, of course, as it should be, since the quaternion field equations from which we start are neither even nor odd with respect to reflections, so that their expression in terms of the *sum* of an even part and an odd part is (in totality) neither even nor odd. Because of the fact that the antisymmetric tensor part of this sum is odd under reflections in space or time, where the Maxwell field equations are also odd under reflections (depending on the *odd* field j_α, as the matter source of the electromagnetic variables), and because the electromagnetic variables themselves are the components of an antisymmetric tensor field, it was found that after taking the covariant divergence of these antisymmetric tensor metric field equations, they then had the same mathematical structure as Maxwell's equations for electromagnetism. That is, they form a vector representation of the Einstein group in terms of field equations whose solutions are an antisymmetric second-rank tensor field.

3. *Astrophysical Applications of the Quaternion Formalism*

The primary interest in this paper is in regard to the astrophysical applications of the generalized field equations (2.7) for general relativity. Thus, the electromagnetic manifestations of interacting charged matter that is incorporated in the predictions of the field equations (2.7) will not be exploited here. Still, the generalized metrical field, q_α, also implies a generalization of the gravitational forces themselves, compared with the predictions of the standard form of the theory of general relativity, in terms of the metric tensor field $g_{\alpha\beta}$. The extra predictions have to do with planetary motion, as well as the motions of stars and other massive bodies, outside of the domain of the solar system. I reported in an earlier paper on the consequences of this generalization in planetary motion problems (Sachs, 1970a).

It was found in the previous analysis of planetary motion, based on the quaternion representation of general relativity, that the extrema of the line integral $\int_{s_1}^{s_2} ds$ yield the same functional expression for the geodesic equation as derived from the usual form of general relativity theory,

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \tag{3.1}$$

However, since the differential element ds is the quaternion $q_\alpha\, dx^\alpha$, rather than a real number, the left-hand side of this equation (and the zero on the right) is also a quaternion, *defining* the derivatives, d/ds, according to Eq. (2.4).

The generalized metrical field q_α contains more information in it than the standard tensor field representation $g_{\alpha\beta}$, as it is a 16-component field rather than a 10-component field, and follows from fully exploiting the symmetry group of general relativity theory. As I have indicated earlier, some of this extra information relates to the "torsion" of space-time. These extra features of space-time appear also in the geodesic equation (3.1) since, in its irreducible expression, this is a two-dimensional Hermitian matrix representation. That is to say, the general form of the geodesic equation (3.1) represents, in the quaternion formalism, four independent equations:

$$\left[\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right]_{ij} = 0 \quad (i, j = 1, 2) \tag{3.1'}$$

for each set of space-time coordinates x^α.

Generally, the *complete* specification of the geodesic path requires the solution of all four of these equations. This conclusion, once again, is a manifestation of the result indicated earlier, that the complete specification of the geodesic curve, according to the quaternion formulation of general relativity theory, requires four parameters, rather than the single-parameter curve of the usual (real number) formulation.

Nevertheless, for the purpose of investigating the implications of the field equations that do not entail torsion effects, a more restricted form of the geodesic equation (3.1) may be used by taking the determinant of the matrix form of this equation. Further, by using the coordinate frame in which ds relates to a time measure in the frame of the object that would move along a geodesic, the geodesic equation further reduces to the form (Sachs, 1970a)

$$\ddot{x}^\gamma + \Gamma^\gamma_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\tfrac{1}{2}Q^{-1}\dot{Q}\,\dot{x}^\gamma \tag{3.2}$$

where $Q = |q_0|^{-2}$, the vertical bars denote the determinant, and the dot refers to the rate of change of the global variables with respect to the time coordinate of a star, as determined from a fixed frame, say that of the Earth observer.

In some well known astrophysical applications, such as the predictions of planetary motion from the Schwarzschild solutions of Einstein's equations, the assumption that the metric tensor $g_{\alpha\beta}$ is independent of the time parameter works well in reproducing the data for these phenomena. Nevertheless, even in this case the metric field q_α could be time dependent in a phase factor, while $g_{\alpha\beta}$ is the time independent. For the relation between $g_{\alpha\beta}$ and the bilinear product (2.5) of quaternion and conjugate quaternion variables shows that the phase factors would cancel out in this form. But the quaternion variable, Q, that appears on the right-hand side of the geodesic equation (3.2), would still be nonzero. Thus, in this case, the predictions of this equation would generally be different from those of the standard form of the geodesic equation, where there is a zero on the right. One of the extra predictions that followed in this way from the earlier analysis (Sachs, 1970a) was a different expression for the angular momenta of planets, while the same result was found for the numerical prediction of the aperiodic contribution to Mercury's perehelion precession, as compared with Einstein's prediction—at least to the order of approximation that was used, according to the accuracy of the measurements compared.

The Hubble Law

In the conventional treatment of the mathematical representations for the Hubble law, such as that of Robertson and Walker (Adler, Bazin, and Schiffer, 1965), conditions are imposed on the metric tensor $g_{\alpha\beta}$ that will lead to a reproduction of the linear Hubble law. With the proper choice of integration constants in the expression of $g_{\alpha\beta}$, as a solution (of a restricted form) of Einstein's equations (2.2), these conditions then lead to a universal global time coordinate (with the tensor component $g_{00} = 1$ in all coordinate frames) and an isotropic space which depends on a time-dependent factor, exp (Ht). This form of the metric tensor then reproduces the Hubble law, $\dot{R} = HR$, where R is taken to be the distance to a star that is moving out in the "exploding universe," relative to the origin of the "big bang," H is Hubble's constant; it appears in this derivation as a constant of the integration.

A difficulty with this means of describing the Hubble law, according to the theory of general relativity, is that it in noncovariant. It is a description that revives the Newtonian concept of absolute time–where the time measure, which is defined in terms of the coordinate along which the universe is evolving, is the same from all reference frames. A second difficulty is that such a form for $g_{\alpha\beta}$ was not in fact derived from the field equations, or from equations of motion. It was rather set up in accordance with symmetry considerations, as well as imposing some extra mathematical conditions that would ensure a form that would predict the Hubble law. Such a form for $g_{\alpha\beta}$ then is forced into the form of Einstein's field equations, at the expense of sacrificing the covariance of these equations. It seems to me, then, that describing the Hubble law in this way cannot yet claim to be a successful prediction of the theory of general relativity itself, even though it may be an empirically correct description of the motions of the stars.

As I have indicated in the Introduction, the aim in this paper is to see if the Hubble law, in particular–which makes a definite claim about the type of motion of the observed stars of the universe–can be *derived from* equations of motion of general relativity theory (even if particular mathematical approximations must be used in carrying out such derivation) based on general physical considerations.

The equation of motion for a star, as a "test body," moving throughout the universe under the influence of the rest of the matter of the universe (in principle), will be taken to be the geodesic equation (3.2). In using this equation of motion, the effects of torsion of the space-time are being ignored at this stage. As a first physical assumption, the velocity of the star will be taken to be very small compared with the velocity of light. In this case, $\dot{x}^0 = c$, $\ddot{x}^0 = 0$. Thus, according to Eq. (3.2),

$$\Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\tfrac{1}{2}Q^{-1}\dot{Q}c \tag{3.3}$$

The geodesic equation (3.2) then takes the form

$$\ddot{x}^k = \Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta(\dot{x}^k/c) - \Gamma^k_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{3.4}$$

for the three spatial coordinates of the observed star, with $k = 1, 2, 3$.

A second physical assumption is that under the actual conditions where the observations of the motions of the stars verify the Hubble law, the effective "force" that acts on the star to cause its motion is time independent. It is due to the average background matter of the rest of the universe, which, in turn, varies sufficiently slowly with time to neglect this dependence. This effective "force," according to general relativity theory, is represented by the geometrical properties of space-time, explicitly in the form of the affine connection terms in the geodesic equations (3.2), (3.4).

With the assumption, then, that the four terms $\Gamma^{\gamma}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$ (for $\gamma = 0, 1, 2, 3$) are time independent in Eq. (3.4), this equation is readily integrated, yielding the solution

$$x^k = K_1 + K_2 \exp(c^{-1}\Gamma^0_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}t) + (\Gamma^k_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}/\Gamma^0_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})ct \quad (3.5)$$

where K_1 and K_2 are the two integration constants, to be determined from the boundary conditions.

For the boundary conditions imposed on the equation of motion (3.2), I will assume the validity of the oscillating universe cosmology, as referred to earlier in the Introduction. This is based on the model in which the presently exploding phase is a cycle that was preceded by, and will be succeeded by, an imploding cycle, continuing in periodic fashion into the indefinite past and future. Since the velocity of the matter in motion changes its direction from the imploding to the exploding phase, there is an inflection point in $\dot{x}^k$ at the times of alternation between implosion and explosion. Calling $t = 0$ the time when the presently observed exploding cycle began, the boundary conditions imposed by this cosmological model are

$$x^k(0) = \dot{x}^k(0) = 0 \quad (3.6)$$

where the star's location at the beginning of this explosion phase is taken to be the origin of the coordinate frame. Note that the formalism is still covariant. and these definitions are with respect to another coordinate frame—that of the astronomer here on Earth.

With the boundary conditions (3.6) in the solutions of the equations of motion (3.5), the two integration constants are found to be as follows:

$$K_1 = -K_2 = c^2 [(\Gamma^k_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})/(\Gamma^0_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})^2]_0 \quad (3.7a)$$

The subscript 0 refers to the values in the bracket at the beginning of the present explosion cycle. With the assumption that $\dot{x}^k \ll c$ [which was used in arriving at Eq. (3.4) for the equation of motion] it follows that the integration constants can be approximated by the form

$$K_1 = -K_2 = [(\Gamma^k_{00})/(\Gamma^0_{00})^2]_0 \quad (3.7b)$$

The solution (3.5) then takes the following explicit form:

$$x^k = [\Gamma^k_{00}/(\Gamma^0_{00})^2]_0[1 - \exp(c^{-1}\Gamma^0_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}t)] + (\Gamma^k_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}/\Gamma^0_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})ct \quad (3.8)$$

Consider the two terms on the right-hand side of Eq. (3.8) separately. The coefficient in front of the first term,

$$[\Gamma^k_{00}/(\Gamma^0_{00})^2]_0 \equiv R^k \tag{3.9}$$

which has the dimension of length, is a function of the spatial coordinates. When describing the furthermost visible stars, this term might be interpreted as the "radius of the universe." When this function is large and spacelike, compared with the separation, ct, the second term on the right-hand side of Eq. (3.8) may be ignored in comparison with the first. This approximation corresponds to the assumption that the distance from the observer to the star is large in comparison with the distance traveled by light from one time that the astronomer observes this star to the next.

With this assumption, that the second term on the right-hand side of Eq. (3.8) can be neglected compared with the first, the solution predicts that the velocity of the star in any of the three spatial directions, x^k, is

$$\dot{x}^k = (c^{-1}\Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta)(x^k - R^k) \tag{3.10}$$

Since R^k cancels in the comparison of two velocities at any two times, Eq. (3.10) then predicts the Hubble law. This law, of course, is verified experimentally by measuring a linear relation between the cosmological red shift in the light emitted by the distant stars and the distance from these stars, and then identifying this red shift with the velocity of the star that is in accordance with the Doppler effect (Tolman, 1934; Sandage, 1972a, b).

It is significant in this analysis that the derivation of the Hubble law is dependent on the appearance of the extra term $\Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta(\dot{x}^k/c)$ in the geodesic equation for the star's coordinates x^k—a term that does not appear in the conventional expression of the geodesic equation in the tensor theory. This extra term corresponds, in the quaternion notation, to the following form for "Hubble's constant":

$$H = c^{-1}\Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -\tfrac{1}{2}Q^{-1}\dot{Q} \tag{3.11}$$

The Q field is defined in Eq. (3.2) in terms of the time component q_0 of the quaternion metrical field solution q_α of the factorized equations (2.7) for general relativity.

With this definition of "Hubble's constant," it is time independent (within the approximations used in this analysis), but it is generally dependent on the spatial coordinates of the star, relative to the location of the initial motion (i.e., when the "big bang" occurred)—from the view of the astronomer here on Earth. That such spatial variation has never been conclusively verified in observations means that the magnitude of H, over the distances traversed by the stars during the time that our astronomers have been recording the motions of the stars, is negligibly small in comparison with the change in H since the time that the "big bang" happened, in this cycle. Still, it is possible that recently observed deviations from the linear Hubble law in the motions of quasars (Sandage, 1972b), is indeed an experimental observation of the deviation of H from constancy—though this possibility must not be taken at this time as more than conjectural, since the data relating to the quasars are still quite controversial among astronomers.

4. *The Spiral Structures of Galaxies*

If the solution (3.8) for the equation of motion of a star is generally valid under the conditions that were imposed in arriving at this solution from the general form (3.2) of the geodesic equation, then it should apply toward the explanation of other stellar phenomena in the night sky that are in addition to the Hubble law. One such phenomenon that still remains unaccounted for by astrophysicists is the spiral, two-dimensional structure of many of the galaxies, including our own Milky Way. I am not referring in this application to the evolution of the matter distribution of the entire universe, as I was in the preceding analysis that led to the Hubble law. I am assuming, rather, that after the explosion (i.e., the "big bang") of this (or any other) cycle of the oscillating universe, as the nuclear matter moved out, ever decreasing its density, globules of stellar matter were formed from a rather homogeneous cosmic dust, leading in turn to the formation of clusters of galaxies, and subglobules of these were formed, leading to the formations of the galaxies themselves. This view is consistent with the model of the evolution of the universe that succeeded the "big bang," according to many of the contemporary astrophysicists.

However, if one should assume that the forces involved in the latter stage, when the galaxies were formed, are the same types of forces that are responsible for the oscillatory dynamics of the entire universe, then it might be assumed, because of the much smaller amount of mass of the galaxy, that over a much smaller time scale compared with the period of oscillation of the entire universe, the matter of the individual galaxies themselves would be successively exploding and imploding in periodic fashion. This conjecture is not unlike the claim in the early days of atomic physics, starting with Bohr's model for hydrogen—and proceeding to the modern quantum mechanical theory of hydrogen—that the same type of force law which is responsible for the attraction of two oppositely charged, macroscopic pith balls (Coulomb's law) should also be responsible for the electron-proton attraction. In this case, also, the time scales involved in the analyses of trajectories of the respective macroscopic and microscopic quantities of interacting matter are different by large orders of magnitude.

With this assumption about the internal dynamics of a galaxy, the same boundary conditions (3.6) should apply to the motion of a star within its own galaxy—the zero time now referring to the beginning of an implosion of the galaxy. In this case, the "radius" of the system considered [Eq. (3.9)] could be of the same order of magnitude as the distance (ct); thus, one cannot neglect the second term on the right-hand side of Eq. (3.8) compared with the first.

In this case, however, we can consider the motion of the star—the "test body"—for time intervals that are small compared with the time it would take light to propagate across the entire domain of the host galaxy. Under these conditions, and assuming that the speed of the star is small compared with the speed of light, the exponential function in the solution (3.8) depends on an argument that is small compared with unity, i.e.,

$$c^{-1}\Gamma^0_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta t \sim \Gamma^0_{00}ct \sim ct/R_G \ll 1 \tag{4.1}$$

where R_G is the order of magnitude of the dimension of the galaxy, as derived from the relation between the affine connection and the "force field" of matter, according to general relativity theory.

With this approximation, the exponential term in (3.8) may be expanded. Keeping only the first two terms, the following expression for the solution of the geodesic equation results:

$$x^k = (ct)\{\Gamma^k_{00}/(\Gamma^0_{00})^2 - [\Gamma^k_{00}/(\Gamma^0_{00})^2]_0\}\Gamma^0_{00} \tag{4.2}$$

With the general expression for the affine connection components in terms of the metric tensor,

$$\Gamma^\gamma_{\alpha\beta} = \tfrac{1}{2}g^{\gamma\lambda}(\partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}) \tag{4.3}$$

and the assumption that $g_{\alpha\beta}$ is an analytic function of the time coordinate, thereby allowing the Taylor expansion,

$$g_{\alpha\beta}(t) = g_{\alpha\beta}(0) + \dot{g}_{\alpha\beta}(0)t + \tfrac{1}{2}\ddot{g}_{\alpha\beta}(0)t^2 + \cdots \tag{4.4}$$

it follows that for small t, the spatial coordinates of the star take the following form:

$$x^k = \tfrac{1}{2}a^k t^2 + b^k t^3 \tag{4.5}$$

where

$$a^k = 4R_G{}^k H_G(\beta^k/\alpha^k - \beta^0/\alpha^0) \tag{4.6a}$$

$$b^k = R_G{}^k H_G(\beta^0/\alpha^0)(2\beta^k/\alpha^k - \beta^0/\alpha^0) \tag{4.6b}$$

$$\beta^k = c[(g^{k0}\ddot{g}_{00} + \dot{g}^{k0}\dot{g}_{00}) + 2(g^{kj}\ddot{g}_{j0} + \dot{g}^{kj}\dot{g}_{j0})]_0 \tag{4.6c}$$

$$\alpha^k = 2c(g^{k0}\dot{g}_{00} + 2g^{kj}\dot{g}_{j0})_0 \tag{4.6d}$$

$$\alpha^0 = (g^{00}\dot{g}_{00}), \qquad \beta^0 = (\dot{g}^{00}\dot{g}_{00} + g^{00}\ddot{g}_{00})_0 \tag{4.6e}$$

The radius $R_G{}^k$ is defined in Eq. (3.9), and the constant H_G is defined in Eq. (3.11), except that the geometrical variables in these equations refer here to the single galaxy. According to the approximations that have been used, $R_G{}^k$ is equal to the ratio $[\alpha^k/(\alpha^0)^2]_0$, and H_G is equal to $\tfrac{1}{2}\alpha^0$.

According to the solution (4.5) for the coordinates of the star in the galaxy, its acceleration in each of the orthogonal spatial directions is

$$\ddot{x}^k = a^k + 6b^k t \tag{4.7}$$

where the coefficients a^k, b^k are time-independent functions defined in Eq. (4.6). Substituting the variables $\ddot{\zeta}^k = \ddot{x}^k - a^k$, Eq. (4.7) can be expressed in the form

$$(\ddot{\zeta}^1)^2 + (\ddot{\zeta}^2)^2 + (\ddot{\zeta}^3)^2 = 36[(b^1)^2 + (b^2)^2 + (b^3)^2]\,t^2 \qquad (4.8)$$

The quaternion metric field q_α for space-time is a four-vector field, and algebraically a second-rank spinor field, of the type $\eta \otimes \eta^*$. With the latter feature in the underlying metric, there is the implication of a "torsion" of space-time. Thus, the dynamics, associated most primitively with the quaternion field for the exploding-imploding stellar systems, implies that, in addition to the translatory motion acquired in the explosion, the star also gains angular momentum, due to an imposed torque. Thus, the entire galaxy would be set into rotational motion in a plane that is perpendicular to the orientation of the initially imposed torque during the explosion. This implies that if x^3 is (defined to be) the axis of rotation, then the quaternion field equations predict that $(b^1, b^2) \gg b^3$. Such rotational motion, which is consistent with the astronomical data in regard to the spiral galaxies, would then be characterized by the equations of motion

$$(\ddot{\zeta}^x)^2 + (\ddot{\zeta}^y)^2 = A^2 t^2 \qquad (4.9)$$

where

$$A^2 = 36[(b^x)^2 + (b^y)^2]$$

and the indices (1, 2) are referred to here as (x, y), denoting the plane of the rotational motion.

The solutions of the equations of motion (4.9) that satisfy the boundary conditions (3.6) predict a spiral motion—the "Cornu spiral"—expressed explicitly in terms of the *Fresnel integrals*:

$$\zeta^x(t) = c\{\textstyle\int_0^t \cos[(A/2c)\tau^2]\,d\tau - t\}, \qquad \zeta^y(t) = c\int_0^t \sin[(A/2c)\tau^2]\,d\tau \qquad (4.10)$$

Thus, the total solution compatible with the boundary conditions of an exploding-imploding oscillatory model is

$$x(t) = \zeta^x(t) + \tfrac{1}{2}a^x t^2, \qquad y(t) = \zeta^y(t) + \tfrac{1}{2}a^y t^2 \qquad (4.11)$$

This solution then describes a superposition of a spiral motion in a two-dimensional plane, characterized by the Fresnel integrals (4.10), and the constant acceleration of the galaxy as a whole, relative to the observer's frame of reference. The constant acceleration part of the motion follows in this analysis from the initial assumption that, at the beginning of an imploding phase of the motion, the geometrical fields that correspond to the Newtonian gravitational force of the classical theory are spatially constant at the early times and are time-independent. This is similar to the classical approximation that predicts a constant acceleration for a body that falls freely toward the earth near its surface.

The actual observation of the internal structure of the spiral galaxies, from our frame of reference, is then a view of the superposition of the spiral paths of the constituent stars. The appearance of different "arms" of the galaxy would then be due to the displacement of starting times of the motions of the different stars, each in turn caused to move spirally toward the hub of its rotating galaxy.

According to the dynamics predicted by this analysis, as time increases the star then moves within the galaxy with ever decreasing radii until it reaches the center of the galaxy—the hub. The physical implication of this type of motion is that one should expect to find the older stars of a galaxy, such as the "red giants," closer to the hub instead of in the outer arms. The details of the shapes of these spirals—the "Cornu spiral"—are given in tables of Fresnel integrals (Jahnke-Emde-Losche, 1960).

According to this dynamical description of the galaxy, then, as time proceeds all of the stars of this system move in a "winding up" phase, toward the hub of the galaxy at its center. This motion is described by the arm of the Cornu spiral curve in the first quadrant of the X-Y plane. This is the implosion phase of the cycle. When a sufficiently great matter density has then built up at the hub, the implosion phase changes into an explosion phase. The dynamics of the star in this phase then follows an unwinding motion from the hub outwards, in a spiral fashion. This path is described by the second part of the Cornu spiral solution, in the third quadrant of the X-Y plane. As the curve passes through the origin of the X-Y plane, the explosion changes to implosion, and the winding up motion of the spiral then starts to repeat itself. In this way, the implosion-explosion behavior of the galaxy of stars continues to evolve harmonically.

5. *Conclusions*

Summing up, a new feature in cosmology that comes from a quaternion representation of general relativity theory is the appearance of phase factors depending on the time coordinate, and in consequence the appearance of extra terms in the geodesic equation to describe the motion of a star. Assuming that (1) the velocity of a star is small compared with the velocity of light, (2) the geometrical terms $\Gamma^{\gamma}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$ in the geodesic equation are time independent at the beginning of any particular exploding phase of the matter of the universe, and in the regions of space near the observed horizon of the universe relative to our position, and (3) the boundary conditions hold that are consistent with a periodic explosion-implosion dynamics of an oscillating universe cosmology, the Hubble law was found to follow from the equations of motion of the star, as a "test body" acted upon by the rest of the matter of the universe. The Hubble constant, which in this theory is found to depend on field variables of the quaternion representation of general relativity theory, is time independent within the approximations used, though dependent on the spatial coordinates. The result implies that possibly the "peculiar" data on quasars are not necessarily due to their anomalous intrinsic properties. They may rather be due to the fact that these particular stellar objects are being observed over a very large distance from us, and the curvature of space-time then effectively alters the measured linear velocity–distance relation.

The spiral structure of galaxies is spelled out in terms of the space dependence of the metrical field variables of this theory. The details of the spiral structures, such as the amplitudes of the spiral arms, must await a more explicit form for the solutions of the field equations. Generally, however, within the approximations used, the shape of these paths within the host galaxy is that of the "Cornu spiral," described mathematically by the Fresnel integrals.

Of course, not all of the galaxies of the night sky are spiral and planar, with a dense hub in the center, as our own galaxy is. Nor are all of the galaxies even planar. According to the theory presented in this paper, the shapes of the planar elliptical galaxies, and those that are not planar, might follow from a breakdown of the boundary conditions and the approximations that were used in this analysis. But, at the present stage, it is felt that the prediction of the spiral shape of some of the galaxies, and the Hubble law, *from equations of motion* associated with the maximally general representation of general relativity theory—that is, the quaternion field representation—does lend further support to the theory of general relativity itself. The analysis also supports the oscillating universe cosmology, and it implies from the predictions of the spiral shapes of rotating galaxies that the constituent galaxies of the universe are similarly in a steady state of periodic exploding and imploding matter, though on a different time scale than the universe as a whole.

The latter conclusion then leads to one further speculation—that the unusually large quantities of energy that have been associated with anomalous stellar objects, and with possible sources of gravitational radiation (Weber, 1969), may in fact relate to the exploding-imploding oscillations of entire galaxies.

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